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COMMENT

Singular solution in a damped double sinh-Gordon system

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Abstract. The singular travelling wave solution of a linearly damped double sinh-Gordon system has been obtained. It is shown that the solution is linearly stable.

The double sinh-Gordon (DShG) system is characterised by the potential (Behera and Khare 1981)

$$V(\varphi) = (b^2/8)\cosh 4\varphi - b\cosh 2\varphi - (b^2/8). \tag{1}$$

$V(\varphi)$ has minima at

$$\varphi = 0 \qquad \text{for } b > 2 \tag{2a}$$

and

$$\cosh 2\varphi = 2/b \qquad \text{for } 0 < b < 2. \tag{2b}$$

For the latter condition there are two degenerate minima. Behera and Khare (1981) have obtained a kink solution for this model. More recently (Joseph and Baby 1983) two more new classes of solitary wave solutions of the DShG system have been obtained. In this comment, following Magyari (1984b) we study analytically the linearly damped DShG system. We find that, contrary to the damped double sine-Gordon (DSG) system (Magyari 1984b) and other damped multistable systems (Magyari 1984a), the damped DShG system may not have a kink solution. Instead it has a singular travelling wave solution which diverges at $\xi(=x-vt)\to 0$, but tends to zero at $\xi\to\pm\infty$. Linear stability analysis around this solution shows that the solution remains stable in the asymptotic limit.

Let us consider a model of a linear chain of particles described by the classical Hamiltonian

$$H = \sum_i \left[\tfrac{1}{2}m\dot{\varphi}_i^2 + \tfrac{1}{2}k(\varphi_{i+1}-\varphi_i)^2 + V(\varphi_i)\right] \tag{3}$$

where m denotes particle mass, k is the strength of the harmonic coupling between neighbouring particles, $\varphi_i(t)$ is the displacement of the ith particle at time t and $V(\varphi_i)$ the on-site potential represented by (1). We also assume that each particle is subjected to a damping force proportional to its velocity. In this way we obtain for the displacement field the equation of motion

$$m\ddot{\varphi} + \eta\dot{\varphi} - kl^2\varphi'' = -\mathrm{d}V/\mathrm{d}\varphi \tag{4}$$

where η is the damping coefficient ($\eta > 0$) and the prime denotes $\partial/\partial x$.

In the absence of dissipation ($\eta = 0$), the kink and solitary wave solutions of the DShG system have been obtained by Behera and Khare (1981) and Joseph and Baby (1983) respectively. Magyari (1984a, b) has shown that, for damped multistable systems, there exists a further localised, linearly stable excitation which is a uniformly driven domain wall (DW). Here we find that for the multistable DShG system such a DW solution may not exist, but there is a singular travelling wave solution, which is linearly stable in the limit $\xi \to \pm\infty$. It is surprising that this particular multistable system (DShG) may not support a DW solution in the presence of damping.

The solution under consideration is a travelling wave governed by the equation

$$W_0\varphi'' + \eta v\varphi' - (b^2/2)\sinh 4\varphi + 2b\sinh 2\varphi = 0 \tag{5}$$

with $W_0 = kl^2 - mv^2 > 0$. There exists a solution to this equation satisfying the boundary condition

$$\varphi(\pm\infty) = 0 \qquad \varphi'(\pm\infty) = 0. \tag{6}$$

It is given by

$$\coth 2\varphi = \cosh(2\xi/\delta) \tag{7}$$

with

$$\delta = \eta v/2b \tag{8}$$

and

$$v = c(1 + \eta^2/8m)^{-1/2} \tag{9}$$

where c is the characteristic velocity of the system.

This solution shows that the DW solution may not exist for $\eta \neq 0$, but instead there exists a singular travelling solution. Such a type of singular solution is not uncommon in non-linear differential equations (Jaworski 1984).

It can be easily proved that (5) has no other finite energy solution subject to the boundary condition (6). This is because, for a potential with doubly degenerate minima (as in (1)), the finite energy solution should move from one minimum of the potential at $\xi = -\infty$ to the adjacent minimum of the potential at $\xi = +\infty$ (Coleman 1977). The boundary condition (6) obviously does not satisfy this condition (2b).

Let us now examine the linear stability and the excitation of the solution (7). For this, we first transform (4) to the comoving frame of the travelling wave solution (7), and then linearise the transformed equation around the solution according to the ansatz

$$\varphi = \varphi(\xi) + \theta(\xi)\exp(-\mathrm{i}\omega t). \tag{10}$$

The transformed equation becomes

$$\theta''(\xi) - (2\mathrm{i}m\omega\eta^{-1} - 1)(\eta v/W_0)\theta'(\xi) + \{m\omega^2 + \mathrm{i}\eta\omega - V''[\varphi(\xi)]\}W_0^{-1}\theta(\xi) = 0 \tag{11}$$

where

$$c^2 = kl^2/m \tag{12}$$

and

$$V''[\varphi(\xi)] = 2b[b + 2b\,\mathrm{cosech}^2(2\xi/\delta) - 2\coth(2\xi/\delta)]. \tag{13}$$

It can be easily checked that (11) has got two solutions corresponding to the frequencies $\omega_1 = 0$ and $\omega_2 = -\mathrm{i}\eta/m$, the solutions for which are given by

$$\theta_1(\xi) = \mathrm{cosech}(2\xi/\delta) \qquad \text{for } \omega = 0 \tag{14a}$$

and

$$\theta_2(\xi) = \exp(4\xi/b\delta)\,\mathrm{cosech}(2\xi/\delta) \qquad \text{for } \omega = -\mathrm{i}\eta/m. \tag{14b}$$

Both of these solutions diverge at $\xi \to 0$. Out of these two solutions only $\theta_1(\xi) \to 0$ as $\xi \to \pm\infty$ and $\theta_2(\infty) \to \infty$ for $b < 2$.

Now we show that the $\omega = 0$ mode is the lowest energy mode. We transform (11) by the ansatz (Magyari 1984b)

$$\theta(\xi) = \exp(-\tfrac{1}{2}\alpha\xi)\psi(\xi) \tag{15}$$

where

$$\alpha = -(4/b\delta)(2\mathrm{i}m\omega\eta^{-1} - 1) \tag{16}$$

to the more familiar Schrödinger-type eigenvalue problem

$$\psi'' + (E - W)\psi = 0 \tag{17}$$

where

$$E = -(4/\delta^2) - (4/b^2\delta^2) + kl^2 W_0^{-2}(m\omega^2 + \mathrm{i}\eta\omega) \tag{18}$$

$$W = (8/\delta^2)\,\mathrm{cosech}^2(2\xi/\delta) - (8/b\delta^2)\coth(2\xi/\delta). \tag{19}$$

Now the third term on the RHS of (18) is always positive or equal to zero (the imaginary part of ω, if any, has to be negative, otherwise (10) will blow up for $t \to \infty$). So the lowest value of E is given by

$$E = -(4/\delta^2) - (4/b^2\delta^2) \tag{20}$$

which corresponds to either $\omega = 0$ or $\omega = -\mathrm{i}\eta/m$. It can be easily checked that, for the lowest eigenvalue (equation (20)), the eigenfunction is given by

$$\psi(\xi) = \exp(2\xi/b\delta)\,\mathrm{cosech}(2\xi/\delta). \tag{21}$$

The eigenmodes corresponding to the frequency $\omega = 0$ and $\omega = -\mathrm{i}\eta/m$ can be obtained by substituting (21) into (15), using (16), and they are the same as given in (14*a*) and (14*b*) respectively. Thus the $\omega = 0$ mode corresponds to the lowest energy.

Thus the solution (7) is linearly stable in the asymptotic limit, as there exists a mode $\theta_1(\xi)$ with frequency $\omega = 0$, such that $\theta_1(\pm\infty) \to 0$. The mode $\theta_2(\xi)$ corresponding to the frequency $\omega = -\mathrm{i}\eta/m$ (the inertia mode as found by Magyari (1984a, b) for the damped multistable systems) is not allowed here, as it diverges in the asymptotic limit.

Thus we conclude that, unlike other damped multistable systems, the damped DShG system may not have a DW solution. Instead it has a singular travelling wave solution. The solution is linearly stable, as the excitation spectrum has got a zero frequency mode which goes to zero in the asymptotic limit. The localised smooth inertia mode, as found in the other damped multistable systems, diverges in the asymptotic limit for this system and hence is not allowed. Finally we say that this type of singular solution lacks direct physical interpretation (Jaworski 1984). However this solution is important analytically because, for want of a finite energy solution with proper boundary condition (at present), this result shows that, for any damped multistable system, the kink solution may not always remain stable.

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References

Behera S N and Khare A 1981 *J. Physique* C **6** 314
Coleman S 1977 *New Phenomena in Subnuclear Physics* ed A Zichichi (New York: Plenum) p 304
Jaworski M 1984 *Phys. Lett.* **100A** 321
Joseph K B and Baby B V 1983 *J. Phys. A: Math. Gen.* **16** 2685
Magyari E 1984a *Phys. Rev. Lett.* **52** 767
—— 1984b *Phys. Rev.* B **29** 7082